BACK PAPER EXAM — ISI BANGALORE — JUNE 2018

INTRODUCTION TO ALGEBRAIC GEOMETRY — 100 POINTS

Notes.
(a) Justify all your steps. Use only those results proved in class.
(b) By default, k denotes an algebraically closed field.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(d) $\mathbb{A}^n_k$ = the affine n-space over k, $\mathbb{P}^n_k$ = the projective n-space over k.
(e) $\mathcal{V}(-)$ = the common zero locus in suitable affine/projective space of a given collection of polynomials, while $\mathcal{I}(-)$ = the ideal of functions vanishing on a given subset of affine/projective space.
(f) $\mathcal{O}(-)$ = the ring of regular functions on a given quasi-projective algebraic set.

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1. [12 points] Define the Krull dimension of a topological space. Explain why $\mathbb{A}^n_k$ has dimension n.

2. [20 points] Let $K \to L$ be an extension of fields.
   (i) Define what it means for elements $a_1, \ldots, a_r \in L$ to be algebraically independent over K.
   (ii) Define the transcendental basis and transcendence degree of L over K.
   (iii) Give an example of a field extension of transcendence degree 1.

3. [16 points] Define the local ring of a point p on a quasi-projective algebraic set Z. Define the Zariski tangent space at that point. Prove that the Zariski tangent space of any point on $\mathbb{A}^n_k$ has vector space dimension n.

4. [20 points] Assume char(k) $\neq 2$. Prove that any irreducible conic in $\mathbb{P}^2_k$ is isomorphic to the curve $\mathcal{V}(X^2 + Y^2 + Z^2)$. Calculate the points at infinity that lie on the projective closure of the following affine conics:

   (a) $\mathcal{V}(x^2 + y^2 - 1)$, (b) $\mathcal{V}(y - x^2)$, (c) $\mathcal{V}(x^2 - y^2 - 1)$.

5. [20 points] For every $a, b \in k$, find the intersection multiplicity at the intersection points of the line $aY = bX$ with the curve $Y^2 = X^2(X + Z)$ in $\mathbb{P}^2$.

6. [12 points] Define what it means for a ring to be a discrete valuation ring. Give an example of such a ring.